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RESEARCH PROJECT INITIATION

Date: September 18, 1973

Project Title: "Kinematics in Fluid Motion"

Project No: E-23-605

Principal Investigator Dr. Andrew W. Harris & Dr. Stephen L. Passman

Sponsor: National Science Foundation

Agreement Period: From 9/1/73 Until 2/29/76

Type Agreement: Grant No. GK-40267

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Reports Required: Annual Progress Letter
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Sponsor Contact Person (s):

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Washington, D. C. 20550

Technical

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Assigned to: Engineering Science & Mechanics

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RESEARCH PROJECT TERMINATION

Date: February 4, 1975

Project Title: Kinematics in Fluid Motion

Project No: E-23-605

Principal Investigator: Dr. A. W. Marris/Dr. S. L. Passman

Sponsor: National Science Foundation

Effective Termination Date: 2/28/75

Clearance of Accounting Charges: 2/29/76

Grant/Contract Closeout Actions Remaining: Final Fiscal Report (OFA)

Assigned to School of Engineering Science & Mechanics

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Other

ENGINEERING COLLEGE

GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL OF ENGINEERING SCIENCE
AND MECHANICS

225 NORTH AVENUE, N.W.
ATLANTA, GEORGIA 30332

January 14, 1976

Dr. George K. Lea
Program Director
Fluid Mechanics Program
Engineering Mechanics Section
Engineering Division
NATIONAL SCIENCE FOUNDATION
Washington, D.C. 20550

Dear Dr. Lea:

I am writing this letter as a closing report on N.S.F. Grant GK-40267.

In addition to the papers that I have already sent to you, there are two papers about to appear:

"On Helical Motions" - Rendiconti Istituto Lombardo

and

"On Complex-Lamellar Motions" - Archive for Rational Mechanics and Analysis.

Reprints for both these papers should be available in a few weeks and I will send copies on to N.S.F. when I have them. Meanwhile I enclose Xerox copies from my galley proofs.

In the paper "On Helical Motions" it is proved in effect that a perfect gas in steady motion <u>cannot</u> describe a helix of any section other than circular section.

A complex-lamellar motion is a motion in which the velocity and vorticity are perpendicular. Plane motions, axi-symmetric motions, and certain helical motions are examples. The paper "On Complex-Lamellar Motions" considers these motions as steady solutions of the Navier-Stokes equations. It is required that the motions be universal, that is to say, the stream-line pattern must be independent of the viscosity. It is shown that beyond the elementary cases mentioned above the <u>only</u> other possibility is a motion whose Lamb surfaces or "energy surfaces" are helicoids, the vortex-lines being circular helices.

For the past few months I have been deeply involved in the problem of universal steady axi-symmetric motions.

In terms of the vorticity, the steady flow Navier-Stokes equations are

$$\operatorname{curl}\,(\underset{\sim}{\omega} \times \underset{\sim}{v}) = -\,\nu\,\operatorname{curl}\,\operatorname{curl}\,\underset{\sim}{\omega} \qquad (1)$$

$\underset{\sim}{v}$ and $\underset{\sim}{\omega}$ being the velocity and the vorticity, and ν being the kinematic viscosity.

Stokes' original solution for the flow past a sphere was based on the setting of the right hand side of (1) zero, and neglecting the left hand side on the basis that the motion was "slow". It is known that if the left hand side of (1) is equal to zero for a steady axi-symmetric motion then the ratio ω/p, where ω is the vorticity magnitude and p is the distance from the axis, bears a constant value on the Lamb surface (i.e., the surface containing the stream-lines and the vortex-lines). This condition was not satisfied in Stokes' solution. There is a semi-trivial class of universal solutions of (1), (i.e., solutions obtained by equating both sides of (1) to zero) for which ω/p is spatially constant. This case is analagous to the case of constant vorticity in plane motion. The question with which I am concerned is: are there any steady universal solutions of (1) for non-rectilinear axi-symmetric motions for which ω/p is not constant?

I conjecture that there are none. I feel that if this conjecture is true then the result is important in that it would show that unsteadiness, three dimensionality, or failure to be universal manifest by a change in stream-line pattern with change in viscosity, would always occur in axi-symmetric motions for which ω/p is not constant.

Seeking to prove the impossibility of motions for which ω/p is not constant I obtained an integral involving the gradient of ω/p normal to the Lamb surface and the four curvatures defining the stream-lines, namely the curvature of the stream surfaces and the surfaces orthogonal to them. By taking the directional derivatives of this condition along and perpendicular to the stream-lines and eliminating variables I hope to show that the curvatures must be constant, and hence that the only solution is the rectilinear flow. The elimination is complicated, one has to establish the independence of two expressions involving the above curvatures. This may be done by setting all but one of the curvatures equal to unity and showing that the eliminant of the last variable does not vanish.

I am in the middle of this calculation at the present time.

Sincerely yours

A. W. Marris
Regents Professor and Co-
Principal Investigator

vc
Enclosures

cc: Dr. M. E. Raville
Director, E.S.M.

Meccanica e Fisica matematica   *Istituto Lombardo (Rend. Sc.)* A 109,   *(1975)*

L

ON HELICA~~N~~ MOTIONS

A. W. MARRIS

Nota presentata dal m. s. Clifford Truesdell
(Adunanza del 21 novembre 1974)

SUNTO. — Si dimostra il teorema seguente riguardo ai flussi stazionari ed isocorici che conservano sa circolazione. Se le superficie di Lamb sono cilindri generali e se le linee di corrente sono eliche, nel caso che queste non sono circolari il flusso si costruisce in base ad un flusso piano mediante la sovrapposizione di un moto uniforme e perpendicolare al detto piano.

1. - Introduction.

In this work I shall be concerned with a class of steady isochoric circulation-preserving motions. An isochoric motion is a motion in which the volume of space occupied by any material region remains constant, however the material region may change its shape. A circulation-preserving motion is a motion possessing an acceleration-potential. If $\mathbf{v}$ is the velocity and $\boldsymbol{\omega} = \text{curl } \mathbf{v}$ is the vorticity, then these conditions are represented respectively by

(1.1) $$\text{div } \mathbf{v} = 0,$$

and

(1.2) $$\mathbf{a} = \boldsymbol{\omega} \times \mathbf{v} + \text{grad } \frac{v_2}{2} = \text{grad } \xi .$$ $\frac{v^2}{2}$

the condition of steadiness being presumed in (1.2). Alternatively the condition (1.2) may be written

(1.3) $$\boldsymbol{\omega} \times \mathbf{v} = \text{grad } \varphi .$$

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Provided the vorticity $\boldsymbol{\omega}$ does not vanish and provided that $\boldsymbol{\omega}$ and $\mathbf{v}$ are not parallel ([1]), it follows from (1.3) that there exist a one-parameter family of surfaces $\varphi(x^\alpha) = \text{constant}$, $\alpha = 1, 2, 3$, containing both the stream-lines and the vortex-lines. These surfaces are called Lamb surfaces.

I shall be interested in the case when the Lamb surfaces are a family of co-axial cylindres of arbitrary cross-section, and the stream-lines are general helices on these surfaces. Such a motion may always be constructed by considering a plane circulation-preserving motion, and superposing a constant velocity perpendicular to the plane of the motion. I shall call this the trivial case and discount it.

cylinders

In this paper I prove the following:

MAIN THEOREM. - *The only steady isochoric circulation-preserving motion admitting Lamb surfaces, whose Lamb surfaces are general cylinders and whose stream-lines are non-trivial helices, is a circular helical motion with circular-cylindrical Lamb surfaces.*

This theorem is purely kinematical in content.

All motions of inviscid incompressible fluids of uniform density under the action of conservative body forces are isochoric and circulations-preserving. Thus, the present theorem requires that the only non-trivial steady helical motions with cylndrical Lamb surfaces, possible for such fluids under conservative body forces, are circular helical motions.

circulation-preserving

cylindrical

2. - Background Material.

The velocity at the point whose co-ordinates are x^α, $\alpha = 1, 2, 3$ is written $\mathbf{v} = v(x^\alpha)\,\mathbf{s}\,(x^\alpha)$, $\mathbf{s}$ being the unit vector tangent to the stream-line. The unit vector $\mathbf{n}\,(x^\alpha)$ points along the principal normal to the stream-line and

$$\mathbf{b} = \mathbf{s} \times \mathbf{n} \tag{2.1}$$

is the unit bi-normal.

([1]) When $\boldsymbol{\omega} \times \mathbf{v} = 0$, the motion is a screw motion, or Beltrami motion [Bjørgum (1951)]. These motions are excluded from the present work.

The vector-lines of $\mathbf{s}$ are postulated to be helices on the general cylindrical Lamb surfaces. They are therefore geodesics on the surfaces. It follows that the unit principal normal $\mathbf{n}$ is also normal to the surface, so that

$$\mathbf{n} = \lambda \operatorname{grad} \varphi , \tag{2.2}$$

where $\varphi(x^a) = \text{constant}$ represents the family of Lamb surfaces. It follows from (2.2) that

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{n} = 0 . \tag{2.3}$$

The vector-lines of $\mathbf{b}$, being the orthogonal trajectories, on the cylindrical Lamb surface, of a family of helices on the surface, are also helices. They are therefore geodesics on the surface, and accordingly their geodesic curvature vanishes ([2]), one has

$$\mathbf{b} \cdot \operatorname{grad} \mathbf{s} \cdot \mathbf{b} = 0 . \tag{2.4}$$

In terms of a basis of Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, with $\mathbf{k}$ pointing along a generator of the cylinder, one may write [Forsyth (1920), p. 23].

$$\begin{aligned} \mathbf{s} &= \cos \alpha \, (-\sin w \, \mathbf{i} + \cos w \, \mathbf{j}) + \sin \alpha \, \mathbf{k} , \\ \mathbf{n} &= -\cos w \, \mathbf{i} - \sin w \, \mathbf{j} , \\ \mathbf{b} &= \sin \alpha \, (\sin w \, \mathbf{i} - \cos w \, \mathbf{j}) + \cos \alpha \, \mathbf{k} . \end{aligned} \tag{2.5}$$

In the relations (2.5) one identifies the unit vector

$$\mathbf{u} = -\sin w \, \mathbf{i} + \cos w \, \mathbf{j} , \tag{2.6}$$

as the unit tangent to the right section of the cylinder. The unit vectors $\mathbf{s}$ and $\mathbf{b}$ are inclined at the angles $\frac{\pi}{2} - \alpha$ and α to the generator of the cylinder. One has

$$\tan \alpha = \frac{\tau}{\varkappa} \tag{2.7}$$

([2]) The cylinders, being developable surfaces, allow two families of geodesics intersecting orthogonally. When the surface is opened out into a plane the two families of helices become families of straight lines intersecting at right angles.

$\varkappa$

where τ and $\varkappa$ are torsion and curvature of the stream-line, and moreover for the general helix (3)

$$\frac{\delta s}{\delta}\left(\frac{\tau}{\varkappa}\right) = 0 \qquad (2.8)$$

$\frac{\delta}{\delta s}\left(\frac{\tau}{\varkappa}\right) = 0$

and the angle α is constant.

It readily follows from (2.3) and $(2.5)^2$, that

$$\frac{\partial w}{\partial z} = 0 \,. \qquad (2.9)$$

With

Whit the simplification afforded by (2.3) and (2.4) one may write [Marris and Wang (1970), Appendix, p. 242].

$$\text{grad}\ \mathbf{s} = \begin{array}{lll} & +\mathbf{s}\mathbf{n}\,\varkappa & \\ & +\mathbf{n}\mathbf{n}\,\psi & +\mathbf{n}\mathbf{b}\,(\Omega-\tau)\,, \\ & -\mathbf{b}\mathbf{n}\,\tau & \end{array} \qquad (2.10)$$

$$\text{grad}\ \mathbf{n} = \begin{array}{lll} -\mathbf{s}\mathbf{s}\,\varkappa & & +\mathbf{s}\mathbf{b}\,\tau \\ -\mathbf{n}\mathbf{s}\,\psi & & -\mathbf{n}\mathbf{b}\ \text{div}\ \mathbf{b}\,, \\ +\mathbf{b}\mathbf{s}\,\tau & & +\mathbf{b}\mathbf{b}\,(\varkappa + \text{div}\ \mathbf{n}) \end{array} \qquad (2.11)$$

$$\text{grad}\ \mathbf{b} = \begin{array}{ll} & -\mathbf{s}\mathbf{n}\,\tau \\ -\mathbf{n}\mathbf{s}\,(\Omega-\tau) & +\mathbf{n}\mathbf{n}\ \text{div}\ \mathbf{b}\,, \\ & -\mathbf{b}\mathbf{n}\,(\varkappa + \text{div}\ \mathbf{n}) \end{array} \qquad (2.12)$$

where, in addition to the parameters already defined,

$$\Omega = \mathbf{s}\cdot\text{curl}\ \mathbf{s} \qquad (2.13)$$

---

(3) I shall mploy the convention $\frac{\delta}{\delta s}, \frac{\delta}{\delta n}, \frac{\delta}{\delta b}, \frac{\delta^2}{\delta s \delta n}$ to denote the quantities $\mathbf{s}\cdot$grad, $\mathbf{n}\cdot$grad, $\mathbf{b}\cdot$grad, $\mathbf{s}\cdot$grad ($\mathbf{n}\cdot$grad), etc. The usual derivative symbol is avoided since these components are anholonomic.

insert dots and bracket.

s·grad

-- s·grad(n·grad)

is the abnormality of the vector field of $\mathbf{s}$, and the parameter ψ is defined by

$$(2.14) \qquad \psi = \mathbf{n} \cdot \operatorname{grad} \mathbf{s} \cdot \mathbf{n} \, .$$

div n The quantity div $\mathbf{s}$ is identified as minus the first curvature of the cylindrical Lamb surface, while $\varkappa + \operatorname{div} \mathbf{n}$ is minus the normal curvature of the $\mathbf{b}$-lines on the surface.

From (2.10), (2.11) and (2.12) one has

$$(2.15) \qquad \operatorname{curl} \mathbf{s} = \operatorname{grad} \times \mathbf{s} = \Omega \mathbf{s} + \varkappa \mathbf{b} \, ,$$

$$(2.16) \qquad \operatorname{curl} \mathbf{n} = \operatorname{grad} \times \mathbf{n} = - \operatorname{div} \mathbf{b} \, \mathbf{s} + \psi \mathbf{b} \, .$$

$$(2.17) \qquad \operatorname{curl} \mathbf{b} = \operatorname{grad} \times \mathbf{b} = (\varkappa + \operatorname{div} \mathbf{n}) \, \mathbf{s} + (\Omega - 2\tau) \, \mathbf{b} \, .$$

The identity

$$\operatorname{curl} \operatorname{grad} F = 0$$

applied to the tensor valued point function F now leads to the commutation formulae

$$(2.18) \qquad \frac{\delta^2 F}{\delta b \, \delta n} - \frac{\delta^2 F}{\delta n \, \delta b} = \Omega \frac{\delta F}{\delta s} - \operatorname{div} \mathbf{b} \frac{\delta F}{\delta n} + (\varkappa + \operatorname{div} \mathbf{n}) \frac{\delta F}{\delta b} \, ,$$

$$(2.19) \qquad \frac{\delta^2 F}{\delta s \, \delta b} - \frac{\delta^2 F}{\delta b \, \delta s} = 0 \, ,$$

$$(2.20) \qquad \frac{\delta^2 F}{\delta n \, \delta s} - \frac{\delta^2 F}{\delta s \, \delta n} = \varkappa \frac{\delta F}{\delta s} + \psi \frac{\delta F}{\delta n} + (\Omega - 2\tau) \frac{\delta F}{\delta b} \, .$$

The relation (2.19) reflects the fact that the $\mathbf{s}$-lines and $\mathbf{b}$-lines are each geodesics on the family of Lamb surfaces.

The parameters occurring in (2.10), (2.11) and (2.12) are related by nine further compatibility conditions. These represent the requirement that the vector field be embedded in a Euclidean space. They may be obtained from the identities $\operatorname{grad} \times \operatorname{grad} \mathbf{s} = 0$, $\operatorname{grad} \times \operatorname{grad} \mathbf{n} = 0$, $\operatorname{grad} \times \operatorname{grad} \mathbf{b} = 0$, or somewhat more easily by lapplying applying (2.18), (2.19) and (2.20) to the base vectors $\mathbf{s}$, and $\mathbf{n}$ and $\mathbf{b}$ in turn.

The required relations are set out in full generality in [Marris and Wang (1970), Appendix]. With the simplification of (2.3) and (2.4) one has

$$\frac{\delta \tau}{\delta n} - \Omega \varkappa + \psi \operatorname{div} \mathbf{b} - (\varkappa + \operatorname{div} \mathbf{n})(\Omega - 2\tau) = 0\,, \tag{2.21}$$

div n

$$\frac{\delta}{\delta s} \operatorname{div} \mathbf{b} - \frac{\delta \psi}{\delta b} = 0\,, \tag{2.22}$$

$-\frac{\delta \psi}{\delta b}$

$$\frac{\delta}{\delta b}(\Omega - \tau) + (\Omega - 2\tau) \operatorname{div} \mathbf{b} + \psi(\varkappa + \operatorname{div} \mathbf{n}) = 0\,, \tag{2.23}$$

$$\frac{\delta \varkappa}{\delta b} + \frac{\delta \tau}{\delta s} = 0\,, \tag{2.24}$$

$$\frac{\delta \tau}{\delta b} - \frac{\delta}{\delta s}(\varkappa + \operatorname{div} \mathbf{n}) = 0\,, \tag{2.25}$$

$$\varkappa(\varkappa + \operatorname{div} \mathbf{n}) + \tau^2 = 0\,, \tag{2.26}$$

$$\frac{\delta \varkappa}{\delta n} - \varkappa^2 - \psi^2 + (2\Omega - 3\tau) = 0\,, \tag{2.27}$$

$\frac{\delta}{\delta n}(\varkappa + \operatorname{div} \mathbf{n})$

$$\frac{\delta}{\delta s}(\varkappa + \operatorname{div} \mathbf{n}) + \frac{\delta}{\delta b} \operatorname{div} \mathbf{b} + (\operatorname{div} \mathbf{b})^2 + (\varkappa + \operatorname{div} \mathbf{n})^2 + \tau(2\Omega - \tau) = 0\,, \tag{2.28}$$

$$\frac{\delta \Omega}{\delta s} + \frac{\delta \varkappa}{\delta b} + \Omega \psi + \varkappa \operatorname{div} \mathbf{b} = 0\,. \tag{2.29}$$

The relations (2.21) to (2.29) include the conditions div curl $\mathbf{s} = 0$, div curl $\mathbf{n} = 0$, div curl $\mathbf{b} = 0$ and the Gauss and Mainardi-Codazzi relations for the Lamb surfaces $\varphi(x^a) = \text{constant}$.

Taking the directional derivative of (2.26) whith respect to $\mathbf{s}$ one obtains

$(\varkappa + \operatorname{div} \mathbf{n}) \frac{\delta \tau}{\delta s}$

$$(\varkappa + \operatorname{div} \mathbf{n}) \frac{\delta \varkappa}{\delta s} + \varkappa \frac{\delta}{\delta s}(\varkappa + \operatorname{div} \mathbf{n}) + 2\tau \frac{\delta \tau}{\delta s} = 0\,. \tag{2.30}$$

From (2.8) one has

$$\varkappa \frac{\delta \tau}{\delta s} - \tau \frac{\delta \varkappa}{\delta s} = 0\,, \tag{2.8}$$

so that

$$\left(\frac{\varkappa}{\tau}(\varkappa + \operatorname{div}\mathbf{n}) + 2\tau\right)\frac{\delta\tau}{\delta s} + \varkappa\frac{\delta}{\delta s}(\varkappa + \operatorname{div}\mathbf{n}) = 0, \tag{2.31}$$

or, by (2.24), (2.25) and (2.26)

$$\varkappa\frac{\delta\tau}{\delta b} - \tau\frac{\delta\varkappa}{\delta b} = 0. \qquad = 0 \tag{2.32}$$

The conditions (2.7), (2.8) and (2.32) show that the helix angle α must bear a constant value on each Lamb surface.

From $(2.5)^2$, one obtains

$$\operatorname{curl}\mathbf{n} = -\left(\cos\frac{\partial w}{\partial x} + \sin w\frac{\partial w}{\partial y}\right)\mathbf{k} \tag{2.33}$$

so that by (2.16) with $(2.5)^1$ and $(2.5)^3$,

$$\operatorname{div}\mathbf{b} = -\mathbf{s}\cdot\operatorname{curl}\mathbf{n} = \sin\alpha\left(\cos w\frac{\partial w}{\partial x} + \sin w\frac{\partial w}{\partial y}\right), \tag{2.34}$$

and

$$\psi = \mathbf{b}\cdot\operatorname{curl}\mathbf{n} = -\cos\alpha\left(\cos w\frac{\partial w}{\partial x} + \sin w\frac{\partial w}{\partial y}\right) \tag{2.35}$$

$\frac{\delta}{\delta n}($

It follows that

$$\operatorname{div}\mathbf{b}\cos\alpha + \psi\sin\alpha = 0, \tag{2.36}$$

or, by (2.8)

$$\varkappa\operatorname{div}\mathbf{b} + \psi\tau = 0. \tag{2.37}$$

The condition (2.36) or its equivalent, condition (2.37), may be looked upon as the geometrical statement that the cylindrical Lamb surfaces are parallel in the direction of the generators. Thus, since

$\varkappa + a$

$$\mathbf{n} = \lambda\operatorname{grad}\varphi \qquad = \lambda\operatorname{grad}p \tag{2.2}$$

one has

$$\operatorname{curl}\mathbf{n} = \operatorname{grad}\lambda\times\operatorname{grad}\varphi = -\mathbf{n}\times\operatorname{grad}\log\lambda,$$

whence, by (2.16)

$$\psi = \frac{\delta}{\delta s}\log\lambda, \qquad \operatorname{div}\mathbf{b} = \frac{\delta}{\delta b}\log\lambda \tag{2.38}$$

The function λ is the distance function for the family of cylinders, and along a curve $\lambda =$ constant on the surface $\varphi(x^\alpha) =$ constant, consecutive surfaces are parallel. One requires that the curve $\lambda =$ constant coincides with a generator. With the notation of equations (2.5) one must have

$$\mathbf{k} \cdot \operatorname{grad} \lambda = \sin \alpha \frac{\delta \lambda}{\delta s} + \cos \alpha \frac{\delta \lambda}{\delta b} = 0 \tag{2.39}$$

The relation (2.36) then follows from (2.38) and (2.39).

3. - Preliminary Analysis.

The vorticity is given by

$$\boldsymbol{\omega} = \operatorname{curl} \mathbf{v} = (\operatorname{grad} v) \times \mathbf{s} + v \operatorname{curl} \mathbf{s},$$

$$= \Omega v \mathbf{s} + \frac{\delta v}{\delta b} \mathbf{n} + \left(\varkappa v - \frac{\delta v}{\delta n}\right) \mathbf{b}, \tag{3.1}$$

$$= \omega_s \mathbf{s} + \omega_n \mathbf{n} + \omega_b \mathbf{b}. \tag{3.2}$$

Since the surfaces $\varphi(x^\alpha) =$ constant whose unit normal is $\mathbf{n}$, are required to be Lamb surfaces, containing both the stream-lines and vortex-lines, one must have

$$\omega_n = \frac{\delta v}{\delta b} = 0. \tag{3.3}$$

From (3.1) and (3.3) one has

$$\boldsymbol{\omega} \times \mathbf{v} = v \omega_b \mathbf{n} \tag{3.4}$$

and

$$v \omega_b \mathbf{n} = \operatorname{grad} \varphi \tag{3.5}$$

For the Lamb surfaces to exist it is evident that the bi-normal component of the vorticity must not vanish.

The condition $\operatorname{curl} (\boldsymbol{\omega} \times \mathbf{v}) = 0$, now yields

$$\operatorname{grad} (v \omega_b) \times \mathbf{n} + v \omega_b \operatorname{curl} \mathbf{n} = 0, \tag{3.6}$$

and by (2.16) and (3.6) one obtains

$$\frac{\delta}{\delta b}\left(v\,\omega_b\right) + v\,\omega_b \operatorname{div} \mathbf{b} = 0, \tag{3.7}$$

$$\frac{\delta}{\delta s}\left(v\,\omega_b\right) + v\,\omega_b\,\psi = 0. \tag{3.8}$$

From (3.3) and (3.7) one has

$$\operatorname{div}(\omega_b\,\mathbf{b}) = \frac{\delta\,\omega_b}{\delta b} + \omega_b \operatorname{div}\mathbf{b} = 0, \tag{3.9}$$

and it follows from (3.2), (3.3) and (3.9) that

$$\operatorname{div}(\omega_s\,\mathbf{s}) = \operatorname{div}(\Omega\,v\,\mathbf{s}) = \operatorname{div}(\Omega\,\mathbf{v}) = 0. \tag{3.10}$$

Since the motion is isochoric,

$$\operatorname{div}\mathbf{v} = \frac{\delta v}{\delta s} + \psi v = 0, \tag{3.11}$$

so that, by (3.10) and (3.11)

$$\frac{\delta\,\Omega}{\delta s} = 0, \tag{3.12}$$

It follows from (3.8) and (3.11) that

$$\frac{\delta\,\omega_b}{\delta s} = 0. \tag{3.13}$$

From the commutation formula (2.19) applied to ω_b and (3.13) one has

$$\frac{\delta^2\,\omega_b}{\delta s\,\delta b} = 0, \tag{3.14}$$

and by (3.9), (3.13) and (3.14), since ω_b cannot vanish,

$$\frac{\delta}{\delta s}\operatorname{div}\mathbf{b} = 0. \tag{3.15}$$

By (2.37)

$$\operatorname{div}\mathbf{b} + \frac{\tau}{\varkappa}\,\psi = 0, \tag{2.37}$$

so that by (3.15) and (2.8)

$$\frac{\delta \psi}{\delta s} = 0 . \tag{3.16}$$

Also by (2.22) and (3.15), one has

$$\frac{\delta \psi}{\delta b} = 0 . \tag{3.17}$$

By (2.32), one has

$$\frac{\delta}{\delta b}\left(\frac{\tau}{\varkappa}\right) = 0 . \tag{2.32}$$

so that by (2.37) and (3.17)

$$\frac{\delta}{\delta b} \operatorname{div} \mathbf{b} = 0 . \tag{3.18}$$

Thus ψ, div $\mathbf{b}$ and $\frac{\tau}{\varkappa}$ each bear constant value on a Lamb surface.

From (2.29) and (3.12)

$$\frac{\delta \varkappa}{\delta b} = -\Omega \psi - \varkappa \operatorname{div} \mathbf{b}$$

so that by (2.37) (4)

$$\frac{\delta \varkappa}{\delta b} = -\psi(\Omega - \tau) . \tag{3.19}$$

By (2.24), one now has

$$\frac{\delta \tau}{\delta s} = \psi(\Omega - \tau) . \tag{3.20}$$

By (2.32), (2.37) and (3.19) there follows

$\frac{\tau}{\varkappa}\frac{\delta \varkappa}{\delta b}$

$$\frac{\delta \tau}{\delta b} = \frac{\tau}{\varkappa} = -\frac{\tau \psi}{\varkappa}(\Omega - \tau) = (\Omega - \tau) \operatorname{div} \mathbf{b} , \tag{3.21}$$

---

(4) It was shown above that the helix angle α is constant on a Lamb surface. By evaluating $\varkappa = \mathbf{b} \cdot \operatorname{curl} \mathbf{s}$ and $\Omega = \mathbf{s} \cdot \operatorname{curl} \mathbf{s}$ from the relations (2.5) and using $\tau = \varkappa \tan \alpha$ one may readily verify that the factor $\Omega - \tau$ occurring in the relations (3.19) to (3.24) is equal t. $\frac{\delta \alpha}{\delta}$.

equal to $\frac{\delta \alpha}{\delta n}$

and

nd by (2.25), one has

$$\frac{\delta}{\delta s}(\varkappa + \operatorname{div} \mathbf{n}) = \frac{\delta \tau}{\delta b} = -\frac{\tau \psi}{\varkappa}(\Omega - \tau) = (\Omega - \tau) \operatorname{div} \mathbf{b}\,. \tag{3.22}$$

By (2.8), one has

$$\frac{\delta \varkappa}{\delta s} = \frac{\varkappa}{\tau}\frac{\delta \tau}{\delta s} = \frac{\varkappa \psi}{\tau}(\Omega - \tau)\,. \tag{3.23}$$

Taking the directional derivative of (2.26) with respect to $\mathbf{b}$ one obtains

$$\frac{\delta}{\delta b}(\varkappa + \operatorname{div} \mathbf{n}) = \frac{\delta}{\delta b}\left(-\frac{\tau^2}{\varkappa}\right),$$

$$= \frac{\tau^2}{\varkappa^2}\frac{\delta \varkappa}{\delta b} - \frac{2\tau}{\varkappa}\frac{\delta \tau}{\delta b},$$

$$= \psi \frac{\tau^2}{\varkappa^2}(\Omega - \tau), \quad \text{by (3.19) and (3.21)}$$

$$= -\frac{\tau}{\varkappa}(\Omega - \tau) \operatorname{div} \mathbf{b}, \quad \text{by (2.37)}\,. \tag{4.23}$$

One may now show hat $\frac{\delta \Omega}{\delta b}$ vanishes so that by (3.12) Ω is constant on a Lamb surface.

By (2.23) and (2.26) one has

$-\tau \operatorname{div} b$

$$\frac{\delta}{\delta b}(\Omega - \tau) + (\Omega - \tau) \operatorname{div} \mathbf{b} - \tau \operatorname{div} \mathbf{b} - \psi \frac{\tau^2}{\varkappa} = 0,$$

so that, by (2.37),

$$\frac{\delta}{\delta b}(\Omega - \tau) = -(\Omega - \tau) \operatorname{div} \mathbf{b}\,.$$

It follows from (3.21) that

$$\frac{\delta \Omega}{\delta b} = 0\,. \tag{3.25}$$

4. - Proof of Main Theorem.

Taking the directional derivative with respect to **b** of the relation

$$(2.21) \qquad \frac{\delta \tau}{\delta n} = \Omega \varkappa - \psi \operatorname{div} \mathbf{b} + (\varkappa + \operatorname{div} \mathbf{n})(\Omega - 2\tau),$$

and using (3.17), (3.18), (3.21), (3.24) and (3.25) one obtains

$$(4.1) \qquad \frac{\delta^2 \tau}{\delta b\, \delta n} = \Omega \frac{\delta \varkappa}{\delta b} - \frac{\tau}{\varkappa} \operatorname{div} \mathbf{b}\, (\Omega - \tau)(\Omega - 2\tau) - $$
$$- 2(\varkappa + \operatorname{div} \mathbf{n})(\Omega - \tau) \operatorname{div} \mathbf{b}\,.$$

Using

$$(2.26) \qquad (\varkappa + \operatorname{div} \mathbf{n}) = - \frac{\tau^2}{\varkappa},$$

and

$$(2.37) \qquad \operatorname{div} \mathbf{b} = - \frac{\psi \tau}{\varkappa},$$

to eliminate $(\varkappa + \operatorname{div} \mathbf{n})$ and $\operatorname{div} \mathbf{b}$ from the right hand side of (4.1), and using the expression (3.19) for $\frac{\delta \varkappa}{\delta b}$, and reducing, one obtains

$$(4.2) \qquad \frac{\delta^2 \tau}{\delta b\, \delta n} = - \psi (\Omega - \tau) \left[\Omega - \frac{\tau^2}{\varkappa^2} (\Omega - 4\tau) \right].$$

Taking the directional derivative with respect to **n** of the relation

$$(3.21) \qquad \frac{\delta \tau}{\delta b} = (\Omega - \tau) \operatorname{div} \mathbf{b}$$

one has

minus

$$(4.3) \qquad \frac{\delta^2 \tau}{\delta n\, \delta b} = (\Omega - \tau) \frac{\delta}{\delta n} \operatorname{div} \mathbf{b} + \operatorname{div} \mathbf{b} \frac{\delta \Omega}{\delta n} - \operatorname{div} \mathbf{b} \frac{\delta \tau}{\delta n}.$$

The commutation formulae (2.18) applied to τ, with (2.26), requires that

$$(4.4) \qquad \frac{\delta^2 \tau}{\delta b\, \delta n} - \frac{\delta^2 \tau}{\delta n\, \delta b} = \Omega \frac{\delta \tau}{\delta s} - \operatorname{div} \mathbf{b} \frac{\delta \tau}{\delta n} - \frac{\tau^2}{\varkappa} \frac{\delta \tau}{\delta b},$$
$$= \psi \Omega (\Omega - \tau) - \operatorname{div} \mathbf{b} \frac{\delta \tau}{\delta n} + \psi \frac{\tau^3}{\varkappa^2} (\Omega - \tau),$$

by (3.20) and (3.21).

$$\left(\psi \frac{\tau^3}{\varkappa^2} (\Omega - \tau) \right.$$

From (4.1), (4.3) and (4.4) one obtains

$$(4.5)\qquad -\psi(\Omega-\tau)\left[2\,\Omega-\frac{\tau^2}{\varkappa^2}(\Omega-5\,\tau)\right]-(\Omega-\tau)\frac{\delta}{\delta n}\operatorname{div}\mathbf{b}-$$

$$-\operatorname{div}\mathbf{b}\frac{\delta\,\Omega}{\delta n}+2\operatorname{div}\mathbf{b}\frac{\delta\tau}{\delta n}=0\,.$$

The relation (4.5) may otherwise be obtained by taking the directional derivative with respect to $\mathbf{s}$ of (2.21), and the directional derivative with respect to $\mathbf{n}$ of (3.20) and using the commutation formula (2.20).

I shall now need to take repeated directional derivatives with respect to $\mathbf{s}$ of the relation (4.5).

By (2.21), (2.26), (3.12), (3.15) and (3.16), one has, after a little reduction,

$$(4.6)\qquad \frac{\delta^2\tau}{\delta s\,\delta n}=\psi(\Omega-\tau)\frac{\varkappa}{\tau}\left[\Omega-\frac{\tau^2}{\varkappa^2}(\Omega-4\,\tau)\right].$$

$-\psi(\Omega-\tau)\frac{\varkappa}{\tau}[\,-$

By (3.15) and (3.18) and the commutation formula (2.20)

$$(4.7)\qquad \frac{\delta^2}{\delta s\,\delta n}\operatorname{div}\mathbf{b}=-\psi\frac{\delta}{\delta n}\operatorname{div}\mathbf{b}\,,$$

and by (3.12), (3.25) and (2.20)

$$(4.8)\qquad \frac{\delta^2\,\Omega}{\delta s\,\delta n}=-\psi\frac{\delta\,\Omega}{\delta n}\,.$$

Taking the directional derivative with respect to $\mathbf{s}$ of (4.5), using (2.8), (2.37), (3.12), (3.15), (3.16), (3.20), (4.6), (4.7) and (4.8) and reducing, one obtains

$$(4.9)\qquad -2\,\psi\frac{\tau^2}{\varkappa^2}(\Omega-\tau)\,(2\,\Omega-\tau)+2\,(\Omega-\tau)\frac{\delta}{\delta n}\operatorname{div}\mathbf{b}+$$

$$+\operatorname{div}\mathbf{b}\frac{\delta\,\Omega}{\delta n}=0\,.$$

The relation (4.9) might also have been obtained by taking the directional derivative of (4.5) with respect to $\mathbf{b}$.

Successive directional derivatives of (4.9) with respect to $\mathbf{s}$, taken in the above manner, lead to the relations

$$(4.10)\qquad -2\psi\frac{\tau^2}{\varkappa^2}(\Omega-\tau)(3\Omega-2\tau)+4(\Omega-\tau)\frac{\delta}{\delta n}\operatorname{div}\mathbf{b}+ + \operatorname{div}\mathbf{b}\frac{\delta\Omega}{\delta n}=0,$$

$$(4.11)\qquad -2\psi\frac{\tau^2}{\varkappa^2}(\Omega-\tau)(5\Omega-4\tau)+8(\Omega-\tau)\frac{\delta}{\delta n}\operatorname{div}\mathbf{b}+ + \operatorname{div}\mathbf{b}\frac{\delta\Omega}{\delta n}=0,$$

and so on.

τ

tau

If $\Omega=T$, it follows from (3.20) and (3.23) that the motion is circular helical. Otherwise the relations (4.9), (4.10), have solution

$$(4.12)\qquad \frac{\delta}{\delta n}\operatorname{div}\mathbf{b}=\psi\frac{\tau^2}{\varkappa^2}(\Omega-\tau).$$

$$(4.13)\qquad \operatorname{div}\mathbf{b}\frac{\delta\Omega}{\delta n}=2\psi\Omega\frac{\tau^2}{\varkappa^2}(\Omega-\tau),$$

and the values (4.12) and (4.13) satisfy (4.11) and succeeding directional derivatives with respect to $\mathbf{s}$ of (4.11).

From (2.21), (2.26) and (2.27),

$$(4.14)\qquad 2\operatorname{div}\mathbf{b}\frac{\delta\tau}{\delta n}=-2\psi\frac{\tau}{\varkappa}\left[\Omega\varkappa+\psi^2\frac{\tau}{\varkappa}-\frac{\tau^2}{\varkappa}(\Omega-2\tau)\right]$$

Substituting the expressions (4.12), (4.13) and (4.14) into (4.5) and reducing, one finally obtains

$$(4.15)\qquad \psi\left\{\left(1+\frac{\tau^2}{\varkappa^2}\right)\Omega^2+\frac{\tau^2}{\varkappa^2}\psi^2\right\}=0.$$

It follows from (4.15) that either $\psi=0$, or else Ω and τ are each zero.

If ψ vanishes it follows from (3.20) and (3.23) that $\varkappa$ and τ are constant along a stream-line, so that the motion is circular-helical.

If Ω and τ are zero, then since rectilinear motions are discounted in the statement of the man theorem, it follows from (2.36) and (2.37)

main

grad $\underline{b}$

that $(\varkappa + \operatorname{div} \mathbf{n})$ and div $\mathbf{b}$ must each be zero. By (2.12) grad $\mathbf{b}$ is zero and the vector-lines of $\underline{b}$ are parallel straight lines. The motion takes place in parallel planes. One has the trivial case discussed in the Introduction.

X

This proves the main theorem.

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REFERENCES

BJORGUM O., 1951. *On Beltrami vector fields and flows*, Part. 1. Universitet I Bergen, Arbok 1951, Naturvitenskapelig rokke nr. 1.

FORSYTH A. R., 1920. *Lectures on the Differential Geometry of Curves and Surfaces*, Cambridge.

MARRIS A. W. and WANG C.-C., 1970. *Solenoidal Screw Fields of Constant Magnitude.* Arch. Rational Mech. Anal. 39, No. 3, P 227-244.

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On Complex-Lamellar Motions

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Introduction

Let $\boldsymbol{v} = v(x^\alpha)\, \boldsymbol{s}(x^\alpha)$, where x^α, $\alpha = 1, 2, 3$ are spatial co-ordinates, be the velocity field for a steady motion, where v is the velocity magnitude and $\boldsymbol{s}$ is the unit vector tangent to the stream-line. The motion is *complex-lamellar* when the vorticity is perpendicular to the velocity, thus

$$\boldsymbol{v} \cdot \boldsymbol{\omega} = 0, \tag{I.1}$$

where

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{v} \tag{I.2}$$

is the vorticity.

My ultimate aim in this work is to delimit the steady rotational universal complex-lamellar motions of Navier-Stokes fluids. These are homogeneous incompressible linearly viscous fluids. Their steady motions satisfy the equations

$$\operatorname{curl}(\boldsymbol{\omega} \times \boldsymbol{v}) = -\nu \operatorname{curl} \operatorname{curl} \boldsymbol{\omega}, \tag{I.3}$$

$$\operatorname{div} \boldsymbol{v} = 0. \tag{I.4}$$

It is required that the complex-lamellar motion be possible for *all* Navier-Stokes fluids: the motion must be independent of the viscosity ν.* Accordingly one must have

$$\operatorname{curl}(\boldsymbol{\omega} \times \boldsymbol{v}) = 0, \tag{I.5}$$

$$\operatorname{curl} \operatorname{curl} \boldsymbol{\omega} = 0. \tag{I.6}$$

The condition (I.4) asserts that the motion is isochoric: the volume of each element of fluid remains constant in time; the condition (I.5) asserts that the steady motion is circulation-preserving: the acceleration is the gradient of a scalar potential.

The problem is thus kinematically defined. One requires to delimit the steady rotational isochoric complex-lamellar circulation-preserving motions whose vorticity satisfies the condition (I.6).

In the interest of generality, and also because it seems to give the most direct approach to the final problem, I look first at the results which are available for steady rotational isochoric complex-lamellar circulation-preserving motions with the condition (I.6) excluded.

* The pressure may depend on ν through the Bernoullian integral of (I.5).

An old conjecture of EARNSHAW (1837) asserts that in a circulation-preserving motion a fluid particle once in complex-lamellar motion would continue in complex-lamellar motion. Considering the class of steady isochoric motions, YIH (1966) showed that a necessary condition for EARNSHAW's conjecture to hold was that the velocity magnitude be constant along a vortex line. He provided a counter-example to show that this condition is not generally satisfied.

Since the vorticity $\boldsymbol{\omega}$ is not to vanish, the condition (I.1) precludes the case in which (I.5) is satisfied identically because $\boldsymbol{\omega} \times \boldsymbol{v} = 0$. It follows that for the motion under consideration these is a non-constant function ϕ such that φ

there

$$\boldsymbol{\omega} \times \boldsymbol{v} = \operatorname{grad} \varphi. \tag{I.7}$$

The family of surfaces $\varphi(x^\alpha) = \text{constant}$ are called Lamb surfaces; they are vector sheets for the stream-lines and the vortex-lines.

One may respond to EARNSHAW's conjecture for steady motions by the following theorem.

Theorem I.1. *For a steady circulation-preserving motion to be complex-lamellar in a neighbourhood, the stream-lines must be geodesics on the Lamb surfaces.*

It is emphasized that this theorem does not require that the motion be isochoric.

I note three particular cases in which this motion may be more completely determined. The first two cases, Theorems I.2 and I.3, have already been proved in the references cited. The third case, Theorem I.4, will be established in this paper.

Theorem I.2 HAMEL-MARRIS-PRIM (1937, 1973, 1952). *Let $\boldsymbol{v}(x^\alpha) = v(x^\alpha)\boldsymbol{s}(x^\alpha)$ be a steady isochoric complex-lamellar circulation-preserving motion such that $\boldsymbol{s} \cdot \operatorname{grad} v = 0$; then the Lamb surfaces must be co-axial circular cylinders and the stream-lines and vortex-lines must be circular helices on these cylinders. In degenerate cases the stream-lines will be concentric circles or parallel straight lines.*

Theorem I.3 (1974), pages 251–255). *Let $\boldsymbol{v}(x^\alpha)$ be a steady non-rectilinear isochoric complex-lamellar circulation-preserving motion such that the torsion of the stream-lines vanishes. The motion must either be a plane motion, so that the Lamb surfaces are cylinders, or else the Lamb surfaces are surfaces of revolution on which the stream-lines are the meridians and the the vortex-lines are the lines of latitude (parallels).*

Theorem I.4. *Let $\boldsymbol{v}(x^\alpha)$ be a steady rectilinear isochoric complex-lamellar circulation-preserving motion. The stream-lines must be parallel straight lines.*

Turning finally to the problem of steady universal complex-lamellar motions of Navier-Stokes fluids, I introduce the kinematical condition (I.6) and delimit these motions as follows.

Theorem I.5. *Let $\boldsymbol{v}(x^\alpha)$ be a steady rotational universal complex-lamellar motion of a Navier-Stokes fluid. The motion must be one of the following:*

1. *Plane or axi-symmetric motion as in Theorem* I.3.
2. *A motion whose stream-lines are parallel straight lines as in Theorem* I.4.

3. *A motion whose Lamb surfaces are general helicoids. The stream-lines are geodesics on the helicoids, while the vortex-lines, the geodesic parallels, are circular helices. The stream-lines are normal to a family of helicoids. The surfaces of constant vorticity are circular cylinders whose axis is the axis of the helicoids. The vorticity magnitude is inversely proportional to the square root of the stream-line torsion.*

I note that the Lamb surfaces of the motion of Theorem I.5, Part 3, cannot be *right* helicoids. On a right helicoid the geodesics orthogonal to the circular helices are straight lines. According to Theorem I.4 these would have to be parallel straight lines.

The circular-helical motion of Theorem I.2 is a special case of the motion of Theorem I.5, Part 3. The universal motion of a Navier-Stokes fluid of this class is obtained from the circular helical motion of STRAKHOVITCH (1963, page 102), by adjusting the arbitrary constants so as to render the motion complex-lamellar. The surfaces orthogonal to the stream-lines are right helicoids.

Examples of the rotationally symmetric motions of Theorem I.3 are HILL's spherical vortex and the finite vortex rings whose existence is established in the recent beautiful analysis of FRAENKEL & BERGER (1974). NORBURY (1972) has given a constructive existence proof of steady vortex rings close to HILL's vortex.

1. Background Material. Proof of Theorem I.1

Is this Section I assemble the basic formulae for the steady isochoric complex-lamellar motion. The motion is determined by the conditions

$$\boldsymbol{v} \cdot \boldsymbol{\omega} = 0, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.2}$$

and

$$\operatorname{curl}(\boldsymbol{\omega} \times \boldsymbol{v}) = 0. \tag{1.3}$$

Equivalent to (1.1) is the condition

$$\Omega_s = \boldsymbol{s} \cdot \operatorname{curl} \boldsymbol{s} = 0, \tag{1.4}$$

which in turn is equivalent to the representation

$$\boldsymbol{s} = \lambda \operatorname{grad} \chi, \qquad \lambda = \frac{1}{|\operatorname{grad} \chi|}. \tag{1.5}$$

The relation (1.5) asserts that the stream-lines are orthogonal to a one-parameter family of surfaces $\chi = \text{constant}$.

One defines $\boldsymbol{n}$ and $\boldsymbol{b}$ to be the unit principal normal and unit bi-normal respectively to the stream-line. If the stream-lines are rectilinear $\boldsymbol{n}$ is chosen to be the normal to the Lamb surface, $\varphi = \text{constant}$, given by (1.7).

Using (1.4), one has the representations (1970)

$$\begin{aligned}\operatorname{grad} \boldsymbol{s} = &\qquad\qquad + \boldsymbol{s}\,\boldsymbol{n}\,\kappa \\ &\qquad\qquad + \boldsymbol{n}\,\boldsymbol{n}\,\psi \qquad - \boldsymbol{n}\,\boldsymbol{b}\,(\Omega_n + \tau) \\ &\qquad\qquad - \boldsymbol{b}\,\boldsymbol{n}\,(\Omega_n + \tau) + \boldsymbol{b}\,\boldsymbol{b}\,\theta\end{aligned} \tag{1.6}$$

$$\begin{aligned}\operatorname{grad} \boldsymbol{n} = & - \boldsymbol{s}\,\boldsymbol{s}\,\kappa \qquad\qquad\qquad + \boldsymbol{s}\,\boldsymbol{b}\,\tau \\ & - \boldsymbol{n}\,\boldsymbol{s}\,\psi \qquad\qquad\qquad - \boldsymbol{n}\,\boldsymbol{b} \operatorname{div} \boldsymbol{b} \\ & + \boldsymbol{b}\,\boldsymbol{s}\,(\Omega_n + \tau) \qquad + \boldsymbol{b}\,\boldsymbol{b}\,(\kappa + \operatorname{div} \boldsymbol{n})\end{aligned} \tag{1.7}$$

$$\begin{aligned}\operatorname{grad} \boldsymbol{b} = &\qquad\qquad - \boldsymbol{s}\,\boldsymbol{n}\,\tau \\ & + \boldsymbol{n}\,\boldsymbol{s}\,(\Omega_n + \tau) + \boldsymbol{n}\,\boldsymbol{n} \operatorname{div} \boldsymbol{b} \\ & - \boldsymbol{b}\,\boldsymbol{s}\,\theta \qquad - \boldsymbol{b}\,\boldsymbol{n}\,(\kappa + \operatorname{div} \boldsymbol{n}).\end{aligned} \tag{1.8}$$

In these formulae κ and τ are respectively the curvature and the torsion of the stream-lines, and Ω_n is the abnormality of the vector-line of $\boldsymbol{n}$,

$$\Omega_n = \boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n}. \tag{1.9}$$

The parameters θ and ψ can be considered to be defined by (1.6). From (1.6), (1.7), and (1.8)

$$\operatorname{curl} \boldsymbol{s} = \operatorname{grad} \times \boldsymbol{s} = \kappa\,\boldsymbol{b}, \tag{1.10}$$

$$\operatorname{curl} \boldsymbol{n} = \operatorname{grad} \times \boldsymbol{n} = - \operatorname{div} \boldsymbol{b}\,\boldsymbol{s} + \Omega_n\,\boldsymbol{n} + \psi\,\boldsymbol{b}, \tag{1.11}$$

$$\operatorname{curl} \boldsymbol{b} = \operatorname{grad} \times \boldsymbol{b} = (\kappa + \operatorname{div} \boldsymbol{n})\,\boldsymbol{s} - \theta\,\boldsymbol{n} + \Omega_b\,\boldsymbol{b} \tag{1.12}$$

where Ω_b is the abnormality of the vector-line of $\boldsymbol{b}$,

$$\Omega_b = \boldsymbol{b} \cdot \operatorname{curl} \boldsymbol{b} = -(\Omega_n + 2\,\tau). \tag{1.13}$$

The vorticity is now given by*

$$\begin{aligned}\omega = \operatorname{curl}(v\,\boldsymbol{s}) &= \operatorname{grad} v \times \boldsymbol{s} + v \operatorname{curl} \boldsymbol{s}, \\ &= \frac{\delta v}{\delta b}\,\boldsymbol{n} + \left(\kappa\,v - \frac{\delta v}{\delta n}\right)\boldsymbol{b}, \qquad (1.14) \\ &= \omega_n\,\boldsymbol{n} + \omega_b\,\boldsymbol{b}, \qquad (1.15)\end{aligned}$$

and one has the representation

$$\begin{aligned}\operatorname{curl}(\omega \times \boldsymbol{v}) = & - 2\,\kappa\,v\,\omega_n\,\boldsymbol{s} \\ & + \left[\frac{\delta}{\delta s}(v\,\omega_n) + v\,(\theta\,\omega_n + \Omega_n\,\omega_b)\right]\boldsymbol{n} \\ & + \left[\frac{\delta}{\delta s}(v\,\omega_b) + v\,(\psi\,\omega_b - \Omega_b\,\omega_n)\right]\boldsymbol{b}.\end{aligned} \tag{1.16}$$

---

* I use the notation $\delta/\delta s$, $\delta/\delta n$, $\delta/\delta b$, to denote the components $\boldsymbol{s} \cdot \operatorname{grad}$, $\boldsymbol{n} \cdot \operatorname{grad}$, $\boldsymbol{b} \cdot \operatorname{grad}$. An expression such as $\delta^2 F/\delta s \delta n$ thus denotes $\boldsymbol{s} \cdot \operatorname{grad}(\boldsymbol{n} \cdot \operatorname{grad} F)$. These components being anholonomic, the usual symbol for the derivative should be avoided.

The s-component of the relation (1.3) requires that either $\kappa=0$ or $\omega_n=0$. If the stream-line curvature κ vanishes, the stream-lines are straight lines. They must be godesics on the Lamb surface. In this case the unit vector $\boldsymbol{n}$ is chosen to be normal to the Lamb surfaces $\varphi=$constant. Since the vortex-lines lie on the Lamb surfaces, ω_n is zero. The unit vector $\boldsymbol{b}$ is tangent to the vortex-line. The vortex-lines are geodesic parallels on the Lamb surfaces.

If ω_n vanishes but κ does not vanish, it follows from (1.16) and the requirement $\boldsymbol{n}\cdot\operatorname{curl}(\boldsymbol{\omega}\times\boldsymbol{v})=0$ that

$$\Omega_n\,\omega_b=0. \tag{1.17}$$

If ω_b vanishes, the motion is irrotational; this case is discounted, and one concludes that

$$\Omega_n=0, \tag{1.18}$$

and by (1.15)

$$\boldsymbol{\omega}\times\boldsymbol{v}=v\,\omega_b\,\boldsymbol{n}. \tag{1.19}$$

Since the Lamb surfaces $\varphi(x^\alpha)=$constant are given by

$$\boldsymbol{\omega}\times\boldsymbol{v}=\operatorname{grad}\varphi, \qquad (1.7),\ (1.20)$$

one must have

$$\boldsymbol{n}=\frac{1}{v\,\omega_b}\operatorname{grad}\varphi. \tag{1.21}$$

It appears that the principal normal $\boldsymbol{n}$ to the stream-line is normal to the Lamb surface. It follows once again that the stream-lines are geodesics and the vortex-lines geodesic parallels in the Lamb surfaces. This is then a necessary condition for a steady* circulation-reserving motion to be complex-lamellar in a neighbourhood. The condition (1.4) has not been used in this analysis. The motion need not be isochoric. This establishes Theorem I.1.

Henceforth in the analysis I shall presume the condition

$$\Omega_n=0. \tag{1.18}$$

Furthermore with the principal normal component of the vorticity zero, the absolute value of the binormal component ω_b is the vorticity magnitude of the verticity I shall write

$$\omega_b=\omega, \qquad \boldsymbol{\omega}=\omega\boldsymbol{b}. \tag{1.22}$$

$\underset{\sim}{\omega} = \omega\underset{\sim}{b}$

The condition $\boldsymbol{b}\cdot\operatorname{curl}(\boldsymbol{\omega}\times\boldsymbol{v})=0$ required by (1.3) now gives by (1.16)

$$\frac{\delta}{\delta s}(v\,\omega)+v\,\psi\,\omega=0. \tag{1.23}$$

(1.2)

Introducing the condition that the motion be isochoric, one has by (1.1) and (1.6)

$$\operatorname{div}\boldsymbol{v}=\operatorname{div}(v\,\boldsymbol{s})=\frac{\delta v}{\delta s}+(\theta+\psi)\,v=0. \tag{1.24}$$

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* The condition of steady motion may be replaced by the condition that the vorticity be steady.

$$\frac{\delta \omega}{\delta s} = \theta \omega. \tag{1.25}$$

Again, since the vorticity must be solenoidal, one must have, by (1.22),

$$\operatorname{div} \omega = \frac{\delta \varphi}{\delta b} + \omega \operatorname{div} \boldsymbol{b} = 0. \tag{1.26}$$

To proceed further one requires certain commutation formulae and compatibility conditions for the anholonomic space determined by the vector fields of $\boldsymbol{s}$, $\boldsymbol{n}$, and $\boldsymbol{b}$.

The identity

$$\operatorname{curl} \operatorname{grad} F = 0$$

applied to the tensor point function F, using the simplifying conditions (1.15), (1.13), and (1.18), yields the commutation formulae

$$\frac{\delta^2 F}{\delta b\, \delta n} - \frac{\delta^2 F}{\delta n\, \delta b} = -\operatorname{div} \boldsymbol{b} \frac{\delta F}{\delta n} + (\kappa + \operatorname{div} \boldsymbol{n}) \frac{\delta F}{\delta b}, \tag{1.27}$$

$$\frac{\delta^2 F}{\delta s\, \delta b} - \frac{\delta^2 F}{\delta b\, \delta s} = -\theta \frac{\delta F}{\delta b}, \tag{1.28}$$

$$\frac{\delta^2 F}{\delta n\, \delta s} - \frac{\delta^2 F}{\delta s\, \delta n} = \kappa \frac{\delta F}{\delta s} + \psi \frac{\delta F}{\delta n} - 2\tau \frac{\delta F}{\delta b}. \tag{1.29}$$

In particular, since

$$\omega_n = \frac{\delta v}{\delta b} = 0, \tag{1.30}$$

it follows from (1.28) that

$$\frac{\delta^2 v}{\delta b\, \delta s} = 0,$$

and hence, by (1.24)

$$\frac{\delta}{\delta b} \operatorname{div} \boldsymbol{s} = \frac{\delta}{\delta b} (\theta + \psi) = 0. \tag{1.31}$$

Finally one has the nine compatibility relations obtained from the identities grad × grad $\boldsymbol{s} = 0$, grad × grad $\boldsymbol{n} = 0$, and grad × grad $\boldsymbol{b} = 0$, namely:

$$-\frac{\delta \tau}{\delta n} + \frac{\delta \theta}{\delta b} - 2\tau(\kappa + \operatorname{div} \boldsymbol{n}) + (\theta - \psi) \operatorname{div} \boldsymbol{b} = 0, \tag{1.32}$$

$$\frac{\delta}{\delta s} \operatorname{div} \boldsymbol{b} + \frac{\delta \theta}{\delta b} + \theta \operatorname{div} \boldsymbol{b} = 0, \tag{1.33}$$

$$\frac{\delta \theta}{\delta n} + \frac{\delta \tau}{\delta b} + 2\tau \operatorname{div} \boldsymbol{b} + (\theta - \psi)(\kappa + \operatorname{div} \boldsymbol{n}) = 0, \tag{1.34}$$

$$\frac{\delta}{\delta s} (\kappa + \operatorname{div} \boldsymbol{n}) - \frac{\delta \tau}{\delta b} + \theta(2\kappa + \operatorname{div} \boldsymbol{n}) = 0, \tag{1.35}$$

$$\frac{\delta\kappa}{\delta b}+\kappa\operatorname{div}\boldsymbol{b}=0, \tag{1.36}$$

$$\frac{\delta\tau}{\delta s}+2\theta\tau-\kappa\operatorname{div}\boldsymbol{b}=0, \tag{1.37}$$

$$\frac{\delta\theta}{\delta s}+\theta^2-\kappa(\kappa+\operatorname{div}\boldsymbol{n})-\tau^2=0, \tag{1.38}$$

$$\frac{\delta\kappa}{\delta n}-\frac{\delta\psi}{\delta s}-\kappa^2-\psi^2-3\tau^2=0, \tag{1.39}$$

$$\frac{\delta}{\delta n}(\kappa+\operatorname{div}\boldsymbol{n})+\frac{\delta}{\delta b}\operatorname{div}\boldsymbol{b}+\theta\psi+(\operatorname{div}\boldsymbol{b})^2+(\kappa+\operatorname{div}\boldsymbol{n})^2-\tau^2=0. \tag{1.40}$$

I.2

2. Proof of Theorem I.4. I.3

Referring the reader to the proofs of Theorems 1.2 and 1.3 in the references given with the statements of the theorems, I shall proceed directly to the proof of Theorem I.4. This theorem pertains to the case of rectilinear stream-lines:

$$\kappa=0, \tag{2.1}$$

It follows from (1.10) that

$$\operatorname{curl}\boldsymbol{s}=0 \tag{2.2}$$

and the surfaces $\chi=$constant, orthogonal to the stream-lines, are parallel surfaces.

I recall the relation

$$\frac{\delta}{\delta b}\operatorname{div}\boldsymbol{s}=\frac{\delta}{\delta b}(\theta+\psi)=0, \tag{1.31}$$

which asserts the geometrical condition that the first curvature of a surface $\chi=$constant, is constant along a $\boldsymbol{b}$-line.

It is not difficult to show that the Gaussian curvature of the surface is also constant along a $\boldsymbol{b}$-line.

From (1.39) for $\kappa=0$,

$$\frac{\delta\psi}{\delta s}=-\psi^2-3\tau^2, \tag{2.3}$$

so that, by (1.31),

$$\frac{\delta^2\psi}{\delta b\,\delta s}=-2\psi\frac{\delta\psi}{\delta b}-6\tau\frac{\delta\tau}{\delta b}=2\psi\frac{\delta\theta}{\delta b}-6\tau\frac{\delta\tau}{\delta b}. \tag{2.4}$$

Also by (1.31) and the commutation formula (1.28)

$$\frac{\delta^2}{\delta b\,\delta s}(\theta+\psi)=0, \tag{2.5}$$

so that by (1.38)

$$\frac{\delta^2\psi}{\delta b\,\delta s}=-\frac{\delta^2\theta}{\delta b\,\delta s}=\frac{\delta}{\delta b}[\theta^2-\tau^2]=2\theta\frac{\delta\theta}{\delta b}-2\tau\frac{\delta\tau}{\delta b}. \tag{2.6}$$

From (2.4) and (2.6) one obtains the relation

$$(\psi-\theta)\frac{\delta\theta}{\delta b}-2\tau\frac{\delta\tau}{\delta b}=0. \tag{2.7}$$

The compatibility condition (1.40) is the Gauss equation for the surface $\chi=\text{constant}$. It shows that the Gaussian curvature of the surface is $\theta\psi-\tau^2$. It is evident from (1.31) that (2.7) is the condition that the curvature is constant along a b-line. The surfaces $\chi=\text{constant}$ are Weingarten surfaces, that is, surfaces admitting a functional relation between their first and Gaussian curvatures.

From (1.14), (1.15) and (1.22) one has

$$\omega=-\frac{\delta v}{\delta n}. \tag{2.8}$$

Since

$$\frac{\delta\omega}{\delta s}=\theta\omega, \tag{1.25}$$

it follows from (2.8) that

$$\frac{\delta^2 v}{\delta s\,\delta n}=\theta\frac{\delta v}{\delta n}. \tag{2.9}$$

Since

$$\frac{\delta v}{\delta b}=0 \tag{1.30}$$

the commutation formula (1.29) applied to v gives

$$\frac{\delta^2 v}{\delta n\,\delta s}=(\psi+\theta)\frac{\delta v}{\delta n}=\operatorname{div}\mathbf{s}\frac{\delta v}{\delta n},$$

a result which by (1.24) may be written

$$\frac{\delta}{\delta n}(\operatorname{div}\mathbf{s}\,v)+\operatorname{div}\mathbf{s}\frac{\delta v}{\delta n}=0$$

or

$$\frac{\delta}{\delta n}\operatorname{div}\mathbf{s}+2\operatorname{div}\mathbf{s}\frac{\delta}{\delta n}\log v=0. \tag{2.10}$$

Taking the directional derivative of (2.10) with respect to $\mathbf{s}$ one has

$$\frac{\delta^2}{\delta s\,\delta n}\operatorname{div}\mathbf{s}+2\frac{\delta}{\delta s}\operatorname{div}\mathbf{s}\frac{\delta}{\delta n}\log v+2\operatorname{div}\mathbf{s}\frac{\delta^2}{\delta s\,\delta n}\log v=0. \tag{2.11}$$

Reversing the order of the mixed derivatives of $\operatorname{div}\mathbf{s}$ and $\log v$ by means of the commutation formula (1.29) and remembering that $\dfrac{\delta}{\delta b}\operatorname{div}\mathbf{s}$ and $\dfrac{\delta}{\delta b}\log v$ are both zero, one obtains

$$\frac{\delta^2}{\delta n\,\delta s}\operatorname{div}s-(\psi+2\operatorname{div}s)\frac{\delta}{\delta n}\operatorname{div}s+2\left[\frac{\delta}{\delta s}\operatorname{div}s-\psi\operatorname{div}s\right]\frac{\delta}{\delta n}\log r=0. \tag{2.12}$$

Eliminating $\dfrac{\delta}{\delta n}\log r$ between (2.10) and (2.12) and simplifying, one obtains

$$\operatorname{div}s\frac{\delta^2}{\delta n\,\delta s}\operatorname{div}s-2(\operatorname{div}s)^2\frac{\delta}{\delta n}\operatorname{div}s-\frac{\delta}{\delta s}\operatorname{div}s\frac{\delta}{\delta n}\operatorname{div}s=0. \tag{2.13}$$

Using now the relations

$$\frac{\delta\tau}{\delta s}=-2\theta\tau,\qquad \frac{\delta\theta}{\delta s}=-\theta^2+\tau^2,\qquad \frac{\delta\psi}{\delta s}=-\psi^2-3\tau^2, \tag{2.14}$$

and

$$\frac{\delta}{\delta s}\operatorname{div}s=\frac{\delta}{\delta s}(\theta+\psi)=-\theta^2-\psi^2-2\tau^2, \tag{2.15}$$

obtained from the compatibility conditions (1.37), (1.38) and (1.39), one reduces the expression (2.13) to the form

$$\begin{aligned}&[-(3\theta^2+\psi^2)+2(\tau^2-3\theta\psi)]\frac{\delta\theta}{\delta n}\\&\quad+[-(3\psi^2+\theta^2)+2(\tau^2-3\theta\psi)]\frac{\delta\psi}{\delta n}-4(\theta+\psi)\tau\frac{\delta\tau}{\delta n}=0.\end{aligned} \tag{2.16}$$

It is now necessary to take the directional derivative with respect to s of the relation (2.16). It appears from (1.31) and the commutation formulae (1.29) that
? terms involving $\dfrac{\delta\theta}{\delta b}$ and $\dfrac{\delta\tau}{\delta b}$ may enter in from the expressions for $\dfrac{\delta^2\theta}{\delta s\,\delta n}$, $\dfrac{\delta^2\psi}{\delta s\,\delta n}$, and $\dfrac{\delta^2\tau}{\delta s\,\delta n}$. Thus by (2.14),

$$\frac{\delta^2\theta}{\delta s\,\delta n}=\frac{\delta^2\theta}{\delta n\,\delta s}-\psi\frac{\delta\theta}{\delta n}+2\tau\frac{\delta\theta}{\delta b}=-(2\theta+\psi)\frac{\delta\theta}{\delta n}+2\tau\frac{\delta\tau}{\delta n}+2\tau\frac{\delta\theta}{\delta b}, \tag{2.17}$$

and similarly by (1.31) and (2.14),

$$\frac{\delta^2\psi}{\delta s\,\delta n}=\frac{\delta^2\psi}{\delta n\,\delta s}-\psi\frac{\delta\psi}{\delta n}+2\tau\frac{\delta\psi}{\delta b}=-3\left[\psi\frac{\delta\psi}{\delta n}+2\tau\frac{\delta\tau}{\delta n}\right]-2\tau\frac{\delta\theta}{\delta b} \tag{2.18}$$

and

$$\frac{\delta^2\tau}{\delta s\,\delta n}=\frac{\delta^2\tau}{\delta n\,\delta s}-\psi\frac{\delta\tau}{\delta n}+2\tau\frac{\delta\tau}{\delta b}=-2\tau\frac{\delta\theta}{\delta n}-(2\theta+\psi)\frac{\delta\tau}{\delta n}+2\tau\frac{\delta\tau}{\delta b}. \tag{2.19}$$

The term involving $\dfrac{\delta\theta}{\delta b}$ and $\dfrac{\delta\tau}{\delta b}$ contained in

$$[-(3\theta^2+\psi^2)+2(\tau^2-3\theta\psi)]\frac{\delta^2\theta}{\delta s\,\delta n}$$
$$+[-(3\psi^2+\theta^2)+2(\tau^2-3\theta\psi)]\frac{\delta^2\psi}{\delta s\,\delta n}-4(\theta+\psi)\tau\frac{\delta^2\tau}{\delta s\,\delta n}$$

reduces to

$$4\tau[\psi+\theta]\left[(\psi-\theta)\frac{\delta\theta}{\delta b}-2\tau\frac{\delta\tau}{\delta b}\right], \tag{2.20}$$

which vanishes by virtue of (2.7). Using the expression (2.14) to calculate the directional derivative with respect to s of the relation (2.16), one obtains, after some reduction, the condition

$$\begin{aligned}&[12\theta^3+3\psi^3+7\theta\psi(3\theta+2\psi)+2\tau^2[4\theta+3\psi]]\frac{\delta\theta}{\delta n}\\&+[2\theta^3+15\psi^3+3\theta\psi(3\theta+8\psi)+2\tau^2[4\theta+3\psi]]\frac{\delta\psi}{\delta n}\\&+4(5\theta+6\psi)(\theta+\psi)\tau\frac{\delta\tau}{\delta n}=0.\end{aligned} \tag{2.21}$$

Multiplying (2.16) by $(5\theta+6\psi)$ and adding to (2.21) to eliminate the term involving $\dfrac{\delta\tau}{\delta n}$, one obtains, on reduction,

$$(\theta+\psi)[\theta^2+8\theta\psi+\psi^2-6\tau^2]\frac{\delta}{\delta n}(\theta+\psi)=0. \tag{2.22}$$

If $\theta+\psi$ $(=\operatorname{div} s)$ vanishes, then by (1.24) $\dfrac{v}{\delta s}$ is zero. By an easy special case of Theorem 1.2 (1973) the stream-lines are parallel straight lines. $\dfrac{\delta v}{\delta s}$

If

$$\frac{\delta}{\delta n}(\theta+\psi)=\frac{\delta}{\delta n}\operatorname{div} s=0, \tag{2.23}$$

then by (2.10) either $\operatorname{div} s=0$ or $\dfrac{\delta}{\delta n}\log v=0$. I have just noted that if $\operatorname{div} s$ vanishes the motion is in parallel straight lines. If $\dfrac{\delta}{\delta n}\log v$ is zero, then by (2.8) the motion is irrotational. This case is discounted.

One is left with the possibility

$$\theta^2+8\theta\psi+\psi^2-6\tau^2=0. \tag{2.24}$$

Taking the directional derivative of (2.24) with respect to s and using the relations

(2.14), one obtains

$$\theta^3+\psi^3-(\theta+\psi)\tau^2+4\theta\psi(\theta+\psi)=0. \tag{2.25}$$

Since the vanishing of $\theta+\psi$ offers no exception to the statement of the theorem, one may factor $\theta+\psi$ from (2.25) to obtain

$$\theta^2+\psi^2+3\theta\psi-\tau^2=0. \tag{2.26}$$

From (2.24) and (2.26) one obtains

$$\theta\psi=\tau^2 \tag{2.27}$$

and finally

$$\theta^2+\psi^2+2\tau^2=0. \tag{2.28}$$

It follows from (2.28) that θ, ψ and τ must each be zero, so that by (1.6) and (1.18), since $\kappa=0$,

$$\operatorname{grad} \boldsymbol{s}=0. \tag{2.29}$$

The vector-lines of $\boldsymbol{s}$ are parallel straight lines. This completes the proof of Theorem I.4.

3. Steady Universal Complex-Lamellar Motions of Navier-Stokes Fluids. Proof of Theorem I.5

To consider the class of universal complex-lamellar motions of a Navier-Stokes fluid one must introduce into the analysis the additional condition (I.5):

$$\operatorname{curl}\operatorname{curl}\underset{\sim}{\omega}=0. \tag{3.1}$$

By (1.12), (1.13) and (1.18)

$$\operatorname{curl}\underset{\sim}{b}=(\kappa+\operatorname{div}\boldsymbol{n})\boldsymbol{s}-\theta\boldsymbol{n}-2\tau\boldsymbol{b}, \tag{3.2}$$

so by (1.22) and (1.25) one has

$$\begin{aligned}\operatorname{curl}\underset{\sim}{\omega}&=\operatorname{curl}(\omega\boldsymbol{b})=(\operatorname{grad}\omega)\times\boldsymbol{b}+\omega\operatorname{curl}\boldsymbol{b}\\&=-\omega[\xi\boldsymbol{s}+2\theta\boldsymbol{n}+2\tau\boldsymbol{b}],\end{aligned} \tag{3.3}$$

where

$$\xi=-\left[\frac{\delta}{\delta n}\log\omega+(\kappa+\operatorname{div}\boldsymbol{n})\right]. \tag{3.4}$$

The condition (3.1) with (1.10), (1.11), (1.18), and (3.2) now yields the three scalar relations

$$\frac{\delta}{\delta n}(\tau\omega)-\frac{\delta}{\delta b}(\theta\omega)-[\theta\operatorname{div}\boldsymbol{b}-\tau(\kappa+\operatorname{div}\boldsymbol{n})]\,\omega=0, \tag{3.5}$$

$$\frac{\delta}{\delta b}(\xi\omega)-2\frac{\delta}{\delta s}(\tau\omega)-2\theta\tau\omega=0, \tag{3.6}$$

$$2\frac{\delta}{\delta s}(\theta\omega)-\frac{\delta}{\delta n}(\xi\omega)+[\kappa\xi+2\psi\theta-4\tau^2]\,\omega=0. \tag{3.7}$$

Expanding (3.5) and using (1.26), one obtains

$$\omega\left[\frac{\delta\theta}{\delta b}-\frac{\delta\tau}{\delta n}-\tau(\kappa+\operatorname{div}\boldsymbol{n})\right]-\tau\frac{\delta\omega}{\delta n}=0. \tag{3.8}$$

Eliminating $\dfrac{\delta\theta}{\delta b}-\dfrac{\delta\tau}{\delta n}$ between (1.32) and (3.8), one obtains

$$\tau\left[\frac{\delta}{\delta n}\log\omega-(\kappa+\operatorname{div}\boldsymbol{n})\right]+(\theta-\psi)\operatorname{div}\boldsymbol{b}=0. \tag{3.9}$$

The relation (3.9) may be written in the alternative form

$$\tau\xi=(\theta-\psi)\operatorname{div}\boldsymbol{b}-2(\kappa+\operatorname{div}\boldsymbol{n})\tau, \tag{3.10}$$

where ξ is the function defined by (3.4).

Expanding (3.6) and using (1.25) and (1.26), one obtains

$$-\omega\operatorname{div}\boldsymbol{b}\,\xi+\omega\frac{\delta\xi}{\delta b}-2\frac{\delta\tau}{\delta s}\omega-4\theta\tau\omega=0. \tag{3.11}$$

Eliminating $\dfrac{\delta\tau}{\delta s}$ between (1.37) and (3.11), since ω is to be non-vanishing, one has,

$$\frac{\delta\xi}{\delta b}=\operatorname{div}\boldsymbol{b}(\xi+2\kappa). \tag{3.12}$$

Expanding (3.7), using (1.25), one obtains

$$\left[2\frac{\delta\theta}{\delta s}+\kappa\xi+2\psi\theta-4\tau^2+2\theta^2\right]-\frac{\delta\xi}{\delta n}-\xi\frac{\delta}{\delta n}\log\omega=0. \tag{3.13}$$

$\dfrac{\delta\theta}{\delta s}$

Eliminating $\dfrac{\theta}{\delta s}$ between (1.38) and (3.13) and eliminating $\dfrac{\delta}{\delta n}\log\omega$ in favor of ξ by (3.4), one obtains

$$\frac{\delta\xi}{\delta n}=\xi^2+(2\kappa+\operatorname{div}\boldsymbol{n})\xi+2\kappa(\kappa+\operatorname{div}\boldsymbol{n})-2\tau^2+2\theta\psi. \tag{3.14}$$

$\dfrac{\delta\xi}{\delta s}$

A formula for $\dfrac{\xi}{\delta s}$ is established as follows. Taking the directional derivative of (3.4) with respect to s, one has

$$\frac{\delta^2}{\delta s\,\delta n}\log\omega=-\frac{\delta\xi}{\delta s}-\frac{\delta}{\delta s}(\kappa+\operatorname{div}\boldsymbol{n}), \tag{3.15}$$

while by (1.25)

$$\frac{\delta^2}{\delta n\,\delta s}\log\omega=\frac{\delta\theta}{\delta n}. \tag{3.16}$$

Applying the commutation formula (1.29) to low ω, one obtains, by use of (1.25),

$\log\omega$

(1.26) and (3.4)

$$\frac{\delta\theta}{\delta n}+\frac{\delta\xi}{\delta s}+\frac{\delta}{\delta s}(\kappa+\operatorname{div}\boldsymbol{n})=\kappa\frac{\delta}{\delta s}\log\omega+\psi\frac{\delta}{\delta n}\log\omega-2\tau\frac{\delta}{\delta b}\log\omega$$
$$=\theta\kappa-\psi(\xi+(\kappa+\operatorname{div}\boldsymbol{n}))+2\tau\operatorname{div}\boldsymbol{b}. \tag{3.17}$$

From (1.34) and (1.35)

$$\frac{\delta\theta}{\delta n}+\frac{\delta}{\delta s}(\kappa+\operatorname{div}\boldsymbol{n})+2\tau\operatorname{div}\boldsymbol{b}+\theta\kappa+(2\theta-\psi)(\kappa+\operatorname{div}\boldsymbol{n})=0, \tag{3.18}$$

so that, on eliminating $\dfrac{\delta\theta}{\delta n}+\dfrac{\delta}{\delta s}(\kappa+\operatorname{div}\boldsymbol{n})$ between (3.17) and (3.18)

$$\frac{\delta\xi}{\delta s}=-\psi\xi+4\tau\operatorname{div}\boldsymbol{b}+2\theta\kappa+2(\theta-\psi)(\kappa+\operatorname{div}\boldsymbol{n}). \tag{3.19}$$

Again from (1.26) and (3.4) one has

$$\frac{\delta^2}{\delta n\,\delta b}\log\omega=-\frac{\delta}{\delta n}\operatorname{div}\boldsymbol{b} \tag{3.20}$$

and

$$\frac{\delta^2}{\delta b\,\delta n}\log\omega=-\frac{\delta\xi}{\delta b}-\frac{\delta}{\delta b}(\kappa+\operatorname{div}\boldsymbol{n}). \tag{3.21}$$

Applying the commutation formula (1.27) to $\log\omega$ and using (1.26) and (3.4), one obtains, by use of (3.12)

$$\frac{\delta}{\delta n}\operatorname{div}\boldsymbol{b}-\frac{\delta}{\delta b}(\kappa+\operatorname{div}\boldsymbol{n})=\frac{\delta\xi}{\delta b}+\xi\operatorname{div}\boldsymbol{b}=2\operatorname{div}\boldsymbol{b}(\xi+\kappa). \tag{3.22}$$

We require a relation obtained by taking the directional derivative with respect to $\boldsymbol{b}$ of the compatibility condition

$$\frac{\delta\kappa}{\delta n}-\frac{\delta\psi}{\delta s}-\kappa^2-\psi^2-3\tau^2=0. \tag{1.39}$$

From (1.27) and (1.36)

$$\frac{\delta^2\kappa}{\delta b\,\delta n}=\frac{\delta^2\kappa}{\delta n\,\delta b}-\operatorname{div}\boldsymbol{b}\frac{\delta\kappa}{\delta n}-\kappa(\kappa+\operatorname{div}\boldsymbol{n})\operatorname{div}\boldsymbol{b}$$
$$=-2\operatorname{div}\boldsymbol{b}\frac{\delta\kappa}{\delta n}-\kappa\frac{\delta}{\delta n}\operatorname{div}\boldsymbol{b}-\kappa(\kappa+\operatorname{div}\boldsymbol{n})\operatorname{div}\boldsymbol{b}. \tag{3.23}$$

From (1.28), (1.31), (1.36) and (1.38),

$$\frac{\delta^2\psi}{\delta b\,\delta s}=-\frac{\delta^2\theta}{\delta b\,\delta s}=2\theta\frac{\delta\theta}{\delta b}+\kappa\operatorname{div}\boldsymbol{b}(\kappa+\operatorname{div}\boldsymbol{n})-\kappa\frac{\delta}{\delta b}(\kappa+\operatorname{div}\boldsymbol{n})-2\tau\frac{\delta\tau}{\delta b}. \tag{3.24}$$

Taking the directional derivative of (1.39) with respect to $\boldsymbol{b}$, and remembering that

$$\frac{\delta\psi}{\delta b}=-\frac{\delta\theta}{\delta b}, \tag{1.31}$$

one obtains

$$2(\psi-\theta)\frac{\delta\theta}{\delta b}-4\tau\frac{\delta\tau}{\delta b}-2\operatorname{div}\boldsymbol{b}\frac{\delta\kappa}{\delta n}-2\kappa(\kappa+\operatorname{div}\boldsymbol{n})\operatorname{div}\boldsymbol{b}+2\kappa^2\operatorname{div}\boldsymbol{b}$$
$$+\kappa\left[\frac{\delta}{\delta b}(\kappa+\operatorname{div}\boldsymbol{n})-\frac{\delta}{\delta n}\operatorname{div}\boldsymbol{b}\right]=0.$$

It then follows from (3.22) that

$$(\psi-\theta)\frac{\delta\theta}{\delta b}-2\tau\frac{\delta\tau}{\delta b}-\operatorname{div}\boldsymbol{b}\frac{\delta\kappa}{\delta n}=\kappa\operatorname{div}\boldsymbol{b}\,[\xi+(\kappa+\operatorname{div}\boldsymbol{n})]. \tag{3.25}$$

The relation (3.25) may also be obtained by taking the directional derivative with respect to $\boldsymbol{n}$ of (1.37) and using (3.22).

Two further relations are obtained by applying the commutation formulae (1.27) and (1.28) to the function ξ of (3.4). First, from (1.28) and (3.12)

$$\frac{\delta^2\xi}{\delta s\,\delta b}-\frac{\delta^2\xi}{\delta b\,\delta s}=-\theta\frac{\delta\xi}{\delta b}=-\theta\operatorname{div}\boldsymbol{b}(\xi+2\kappa). \tag{3.26}$$

Again from (1.33), (3.12) and (3.19)

$$\frac{\delta^2\xi}{\delta s\,\delta b}=(\xi+2\kappa)\frac{\delta}{\delta s}\operatorname{div}\boldsymbol{b}+\operatorname{div}\boldsymbol{b}\left[\frac{\delta\xi}{\delta s}+2\frac{\delta\kappa}{\delta s}\right]$$
$$=-(\xi+2\kappa)\left[\frac{\delta\theta}{\delta b}+\theta\operatorname{div}\boldsymbol{b}\right]+\operatorname{div}\boldsymbol{b}\left[2\frac{\delta\kappa}{\delta s}-\psi\xi+4\tau\operatorname{div}\boldsymbol{b}\right. \tag{3.27}$$
$$\left.+2\theta\kappa+2(\theta-\psi)(\kappa+\operatorname{div}\boldsymbol{n})\right].$$

Taking the directional derivative of (3.19) with respect to $\boldsymbol{b}$ and using (1.31) and (3.12), ~~me~~ we have

$$\frac{\delta^2\xi}{\delta b\,\delta s}=(\xi+2\kappa+4(\kappa+\operatorname{div}\boldsymbol{n}))\frac{\delta\theta}{\delta b}-\psi\operatorname{div}\boldsymbol{b}(\xi+2\kappa)+4\left[\operatorname{div}\boldsymbol{b}\frac{\delta\tau}{\delta b}+\tau\frac{\delta}{\delta b}\operatorname{div}\boldsymbol{b}\right] \tag{3.28}$$
$$-2\theta\kappa\operatorname{div}\boldsymbol{b}+2(\theta-\psi)\frac{\delta}{\delta b}(\kappa+\operatorname{div}\boldsymbol{n}).$$

From (3.26), (3.27) and (3.28) one obtains

$$2[\xi+2(2\kappa+\operatorname{div}\boldsymbol{n})]\frac{\delta\theta}{\delta b}-2\operatorname{div}\boldsymbol{b}\frac{\delta\kappa}{\delta s}+4\tau\frac{\delta}{\delta b}\operatorname{div}\boldsymbol{b}+4\operatorname{div}\boldsymbol{b}\frac{\delta\tau}{\delta b}$$
$$+2(\theta-\psi)\frac{\delta}{\delta b}(\kappa+\operatorname{div}\boldsymbol{n})-\operatorname{div}\boldsymbol{b}\left(\frac{\delta\xi}{\delta s}+\psi\xi+2(\theta+\psi)\kappa\right)=0. \tag{3.29}$$

The relation (3.29) may otherwise be obtained by taking the directional derivative with respect to $\boldsymbol{s}$ of (3.22).

and

Second, taking the directional derivative with respect to $\boldsymbol{b}$ of (3.14) using (1.31), (1.36) and using

$$\frac{\delta\psi}{\delta b}=-\frac{\delta\theta}{\delta b} \quad \text{and} \quad \frac{\delta\kappa}{\delta b}=-\kappa \operatorname{div} \boldsymbol{b}, \qquad (1.31), (1.36)$$

one has

$$\begin{aligned}\frac{\delta^2\xi}{\delta b\,\delta n}=&[2\xi+(2\kappa+\operatorname{div}\boldsymbol{n})]\frac{\delta\xi}{\delta b}+(2\kappa+\xi)\frac{\delta}{\delta b}(\kappa+\operatorname{div}\boldsymbol{n})+2(\psi-\theta)\frac{\delta\theta}{\delta b}\\ &-4\tau\frac{\delta\tau}{\delta b}-\kappa\operatorname{div}\boldsymbol{b}\,[\xi+2(\kappa+\operatorname{div}\boldsymbol{n})].\end{aligned} \tag{3.30}$$

From (3.12)

$$\frac{\delta^2\xi}{\delta b\,\delta n}=(2\kappa+\xi)\frac{\delta}{\delta n}\operatorname{div}\boldsymbol{b}+\operatorname{div}\boldsymbol{b}\left(\frac{\delta\xi}{\delta n}+2\frac{\delta\kappa}{\delta n}\right). \tag{3.31}$$

From the commutation formula (1.27) with (3.30) and (3.31) one obtains

$$\begin{aligned}&2\left[(\psi-\theta)\frac{\delta\theta}{\delta b}-2\tau\frac{\delta\tau}{\delta b}-\operatorname{div}\boldsymbol{b}\frac{\delta\kappa}{\delta n}\right]+(2\xi+\kappa)\frac{\delta\xi}{\delta b}\\ &+(2\kappa+\xi)\left[\frac{\delta}{\delta b}(\kappa+\operatorname{div}\boldsymbol{n})-\frac{\delta}{\delta n}\operatorname{div}\boldsymbol{b}\right]-\kappa\operatorname{div}\boldsymbol{b}\,[\xi+2(\kappa+\operatorname{div}\boldsymbol{n})]=0.\end{aligned} \tag{3.32}$$

Substituting into (3.32) the expressions (3.12), (3.22) and (3.25), one obtains

$$\begin{aligned}\operatorname{div}\boldsymbol{b}\,[&2\kappa(\xi+(\kappa+\operatorname{div}\boldsymbol{n}))+(2\xi+\kappa)(\xi+2\kappa)-2(2\kappa+\xi)(\xi+\kappa)\\ &-\kappa(\xi+2(\kappa+\operatorname{div}\boldsymbol{n}))]=0,\end{aligned}$$

which reduces to

$$\kappa^2\operatorname{div}\boldsymbol{b}=0. \tag{3.33}$$

It follows from (3.33) that either κ or div $\boldsymbol{b}$ or both of these quantities must vanish. If κ vanishes, then Theorem 1.4 holds, and the stream-lines must be parallel straight lines.

One is left with the case

$$\operatorname{div}\boldsymbol{b}=0. \tag{3.34}$$

From (3.10) and (3.34)

$$\tau\xi=-2(\kappa+\operatorname{div}\boldsymbol{n})\,\tau. \tag{3.35}$$

If the torsion τ vanishes, one has the possibilities given by Theorem 1.3. The motion is either plane or rotationally symmetric.

One is left with case

$$\xi=-2(\kappa+\operatorname{div}\boldsymbol{n}) \tag{3.36}$$

The expression (3.14) for $\dfrac{\delta\xi}{\delta n}$ now reduces to (1.40) with div $\boldsymbol{b}$ set equal to zero. From (3.12) and (3.36) one has

$$\frac{\delta}{\delta b}(\kappa+\operatorname{div}\boldsymbol{n})=0. \tag{3.37}$$

Substituting (3.34), (3.36) and (3.37) into (3.29), one sees that

$$\kappa \frac{\delta \theta}{\delta b} = 0$$

so that since κ is not to vanish, one must have

$$\frac{\delta \theta}{\delta b} = 0. \tag{3.38}$$

Again from (3.25) and (3.38), since the torsion τ does not vanish, one has

$$\frac{\delta \tau}{\delta b} = 0. \tag{3.39}$$

One verifies from (3.36) and (3.34) that the formula (3.19) for $\frac{\delta \xi}{\delta s}$ becomes the compatibility condition (1.35).
Finally from (1.31) and (1.36) respectively one obtains

$$\frac{\delta \psi}{\delta b} = 0 \tag{3.40}$$

and

$$\frac{\delta \kappa}{\delta b} = 0. \tag{3.41}$$

We conclude that all the vector-field parameters as well as v and ω are constant along a vector-line of $\boldsymbol{b}$.

Writing $-2\tau \boldsymbol{b} + \kappa_b \boldsymbol{b}_b$

$$\operatorname{curl} \boldsymbol{b} = -2\tau \boldsymbol{b} + \kappa_b \boldsymbol{b}_b \tag{3.42}$$

where κ_b is the curvature of the b-line, and $\boldsymbol{b}_b$ is the unit binormal to the b-line, one has

$$\boldsymbol{b}_b = \frac{(\kappa + \operatorname{div} \boldsymbol{n})\, \boldsymbol{s} - \theta\, \boldsymbol{n}}{\kappa_b} \tag{3.43}$$

and

$$\kappa_b = [(\kappa + \operatorname{div} \boldsymbol{n})^2 + \theta^2]^{\frac{1}{2}}. \tag{3.44}$$

Then κ_b is constant along a b-line. Again the principal normal to the b-line is given by

$$\boldsymbol{n}_b = \boldsymbol{b}_b \times \boldsymbol{b} = \frac{-(\kappa + \operatorname{div} \boldsymbol{n})\, \boldsymbol{n} - \theta\, \boldsymbol{s}}{\kappa_b}. \tag{3.45}$$

Applying the Serret-Frénet formula

$$\frac{\delta \boldsymbol{b}_b}{\delta b} = -\tau_b\, \boldsymbol{n}_b$$

one sees that the torsion τ_b of the b-lines is given by

$$\tau_b = -\tau. \tag{3.46}$$

Thus τ_b is constant along a b-line.

The b-lines, which are the vortex-lines, are thus circular helices. From (3.4) and (3.26)

$$\frac{\delta}{\delta n}\log\omega=(\kappa+\operatorname{div}\boldsymbol{n}). \tag{3.47}$$

Also, by (1.25) one has

$$\frac{\delta}{\delta s}\log\omega=\theta \tag{3.48}$$

From (3.43), (3.45), (3.47) and (3.48)

$$\frac{\delta}{\delta b_b}\log\omega=0, \tag{3.49}$$

$$\frac{\delta}{\delta n_b}\log\omega=-\kappa_b. \tag{3.50}$$

The vector lines of $\boldsymbol{b}_b$ lie in the surfaces $\omega=$constant. So, too, do the vector-lines of $\boldsymbol{b}$. The principal normal $\boldsymbol{n}_b$ is normal to the surfaces. The b-lines are thus geodesics on the surfaces $\omega=$constant. The surfaces $\omega=$constant must be co-axial circular cylinders which intersect the Lamb surfaces at a constant angle along the circular helical b-lines (vortex-lines).

The Lamb surfaces, $\varphi=$constant, containing the stream-lines and vortex-lines are general helicoids. The stream-lines are geodesics on these surfaces, and the circular helical vortex-lines are the geodesic parallels on the surface. The surface may be considered to be generated by simultaneously rotating a stream-line about an axis (the axis of the circular cylindrical surfaces $\omega=$constant) and translating it in the direction of the axis with a velocity proportional to the angular velocity of rotation.

It is apparent that the surfaces $\chi=$constant orthogonal to the stream-lines are also general helicoids. The Gauss equation for these surfaces (1.40) exhibits div $\boldsymbol{b}$ as the geodesic curvature of the vector-lines of $\boldsymbol{n}$ on these surfaces. Since div $\boldsymbol{b}$ vanishes the n-lines are geodesics on these surfaces. The circular-helical b-lines (vortex-lines) are geodesic parallels. The directions $\boldsymbol{s}$ and $-\boldsymbol{b}$ being respectively principal normal and bi-normal to the n-line, one verifies that the torsion of an n-line is also τ.* From (1.32) and (1.37) one has

$$\frac{\delta}{\delta n}\log\tau^{\frac{1}{2}}=-(\kappa+\operatorname{div}\boldsymbol{n}), \tag{3.51}$$

$$\frac{\delta}{\delta s}\log\tau^{\frac{1}{2}}=-\theta. \tag{3.52}$$

asymptotic

* The torsions of the asymtotic lines through a point on a surface are equal in magnitude and opposite in sign, as also are the torsions of two orthogonal geodesics. The b-lines and n-lines, being orthogonal, can be asymptotic lines on the surfaces $\chi=$constant only if these surfaces are minimal surfaces, so that div $\boldsymbol{s}=0$. In this case the s-lines and b-lines are orthogonal geodesics on the Lamb surfaces $\varphi=$constant. These surfaces are then circular cylinders while the surfaces $\chi=$constant are right helicoids. Again, the b-lines and n-lines can be orthogonal geodesics on the surfaces $\chi=$constant only if these surfaces are developable surfaces. These surfaces must then be circular cylinders. The Lamb surfaces $\varphi=$constant would be right helicoids. As I remarked in the Introduction, this case is invalidated by Theorem 1.4. sp

so that, by (3.47) and (3.48)

$$\omega \tau^{\frac{1}{2}} = \text{constant}. \tag{3.53}$$

This completes the proof of Theorem 1.5.

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References

1837 EARNSHAW, S., On fluid motion, so far as it is expressed by the equation of continuity. Trans. Cambridge Phil. Soc. **6**, 203–233.

1894 HILL, M.J.M., Phil. Trans. Roy. Soc., London (A) **185**, 213–245.

1937 HAMEL, G., Potentialströmungen mit konstanter Geschwindigkeit. Sitzungsber. preuss. Akad. Wiss., phys.-math. Kl. p. 5–20.

1952 PRIM, R.C., Steady rotational flow of ideal gases. J. Rational Mech. Anal. **1**, 425–497.

1963 BERKER, R., Intégration des équations du mouvement d'un fluide visqueux incompressible. Handbuch der Physik, Band VIII/2. Berlin-Göttingen-Heidelberg: Springer.

1966 YIH, C.-S., On the Earnshaw conjecture. Zeitschrift für angewandte Mathematik und Mechanik **46**, (7), 471–472.

1970 MARRIS, A.W., & C.-C. WANG, Solenoidal screw fields of constant magnitude. Arch. Rational Mech. Anal. **39**, 227–244, Appendix.

1972 NORBURY, J., A steady vortex-ring close to Hill's spherical vortex. Proc. Cambridge Phil. Soc. **72**, 253–284.

1973 MARRIS, A.W., Hamel's theorem. Arch. Rational Mech. Anal. **51**, 85–105.

1974 1. MARRIS, A.W., Steady isochoric circulation-preserving motions whose stream-lines are plane geodesics on the Lamb surfaces. Arch. Rational Mech. Anal. **54**, 238–256.

1974 2. FRAENKEL, L.E., & M.S. BERGER, A global theory of steady vortex rings in an ideal fluid. Acta Mathematica **132**, 13–51.

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January 20, 1976

Dr. George K. Lea
Program Director
Fluid Mechanics Program
Engineering Mechanics Section
Engineering Division
NATIONAL SCIENCE FOUNDATION
Washington, D. C. 20550

Dear Dr. Lea:

This letter is a final report on N.S.F. Grant GK-40267.

The research done is divided into three phases.

The first phase is concerned with research on convected derivatives. It is shown that the definition and basic recurrence relation for the convected derivative of a tensor have wider implications than was formerly suspected. In particular, the recurrence relation may be written as an ordinary differential equation in time. Subject to reasonable initial conditions this equation may be solved generally, and the solution generalizes the usual convection-diffusion theorems for vorticity and deformation rate from fluid mechanics. These simple facts then form the groundwork for a number of significant theorems. For example, Rivlin-Ericksen tensors are shown to be special cases of convected rates, so that their(known) recurrence relation is a special case of the general one. For a motion in which one of the Rivlin-Ericksen tensors, no matter of how high an order, vanishes, it is possible to explicitly construct the motion by solving a set of ordinary differential equations. Viscometric flow is an example of such a case. The case of generalized circulation-preserving motion, which itself subsumes the case of circulation-preserving motion, is subsumed by this case. The major new result here is a nonlinear partial differential equation which is a necessary and sufficient condition that a motion, once circulation-preserving, remains circulation-preserving. There is also application of the general theory to convected stress rates. Here follows the result that if, in some motions, the convected stress rate vanishes for some interval of time, then all of the lower stress rates, and indeed the Cauchy stress iteself, may be determined from their initial values and the present value of the deformation gradient, independent of the constitutive equation of the material. These results have appeared in the Rendiconti of the Istituto Lombardo, but I have not yet received reprints. When they arrive, copies will be sent to N.S.F.

Second, two short works on kinematics of fluids have been written in response to inquiries by Professor Marris. The first, "A Note on Flow Through a Surface," generalizes the known result (whose converse is not true) that the vorticity at a material surface is tangent to the surface to the case of non-material surfaces. The main results are if the velocity field of a flow is normal to a surface, the vorticity field is tangent to that surface, and conversely, if at every point on a surface the vorticity vanishes or is tangent to the surface, the velocity is normal to the surface. These theorems then make the standard results applicable to waves and granular media. This work has been submitted to Zeitschrift für angewandte Mathematik und Mechanik.

The second work addresses the problems of limiting vector lines (for example, limiting stream- or vortex- lines at a boundary). It shows that, for a non-trivial solenoidal field, the limiting vector lines are parallel to the isochoric velocity field of an incompressible fluid and to the vorticity field of any fluid. I have been hesitant for some time to submit this result for publication because it would appear that surely such a simple result must be known. It appears, however, that such is not the case.

The third phase addresses the concept of inertia and balance laws in continuous media. S. C. Cowin and F. Leslie, in a paper which has not yet appeared, show that under a reasonable set of assumptions it is possible to prove that the resultant force on a body is a linear function of the acceleration of its center of mass, f = Ma, where M is a tensor. This is gratifying in that "Newton's second law" is, in a sense, proven, but disturbing in that the force is not necessarily collinear with the acceleration. The implications of this result are investigated fully, it is found, in particular, that such a result does not contradict the known principles of continuum physics, if they are interpreted properly. Sufficient conditions are given so that the usual results are recovered. This work is currently being typed for submission, and research continues on this topic.

Sincerely yours.

S. L. Passman
Associate Professor and
Co-Principal Investigator

vc

cc: Dr. M. E. Raville
Director, E.S.M.